# Mean-Variance Asset Pricing after Variable Taxes[1]

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**Abstract:** Huang and Litzenberger (1988) found that capital market equilibrium does not exist as a priori, but depends on the relationship of two financial parameters: the expected rate of return on the (global) minimum variance portfolio and the risk-free rate (overnight rate). Risk averse investors will undertake risky investments if and only if the former exceeds the latter. The objective of this paper is to derive equilibrium solutions after taxes in circumstances in which this condition is not fulfilled (i.e., equilibrium before taxes does not exist). Therefore, a variable tax on riskless assets is then defined. The result is an asset pricing formula after variable taxes due to the Capital Asset Pricing Model (CAPM), where the tax rate acts as a control variable to ensure equilibrium after taxes and to stabilize financial markets.

**Zusammenfassung:** Ausgangspunkt der Untersuchung bildet eine, von Huang und Litzenberger (1988) formulierte Fallunterscheidung, wonach ein Gleichgewicht auf dem Kapitalmarkt nicht a priori existiert, sondern von der Konstellation zweier finanzwirtschaftlicher Größen abhängt. Diese Fallunterscheidung wird zunächst im Detail dokumentiert und beinhaltet einen Renditevergleich zwischen dem sog. globalen Minimum-Varianz-Portfolio und der risikofreien Anlage. Ziel der weiteren Untersuchung ist es, analytische Gleichgewichtslösungen für den ungünstigen Fall abzuleiten, dass kein Gleichgewicht zustande kommt. Dies erfolgt hier mit Hilfe einer Steuer auf risikofreie Anlagen. Der Steuersatz fungiert als zusätzliche Variable, um ein Gleichgewicht nach Steuern zu konstituieren und den Kapitalmarkt auf diese Weise zu stabilisieren.

**Keywords:** CAPM, equilibrium, capital market, variable tax, controlling

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# 1 Introduction

In a systematic elaboration of mean-variance asset pricing modelling, Huang and Litzenberger (1988) set up an equilibrium condition, which compares two financial parameters: the expected rate of return on the global minimum variance portfolio and the riskless rate. Depending on the relationship of these two parameters, equilibrium on capital markets does or does not exist. If equilibrium does not exist, investors deposit their money into a riskless bank account. In this case, no one undertakes risky investments. Risky assets do not have a positive price, which is by definition, a precondition for equilibrium. If equilibrium does not exist, then a pricing formula for risky assets according to the Capital Asset Pricing Model (CAPM) does not exist. Because Huang and Litzenberger (1988) were the first to formulate this condition explicitly, it is called the Huang-Litzenberger condition. Thus far, this condition has only been mentioned now and then in a footnote, for example by Kandel and Stambaugh (1987). The real importance of the Huang-Litzenberger condition for mean-variance asset pricing has not been noticed to this day.

The objective of this paper is to deduce equilibrium solutions for asset pricing after taxes, in case the Huang-Litzenberger condition is not fulfilled. The question is:

> Is it possible to set up a favourable tax system in order to ensure equilibrium after taxes and to stabilize financial markets?

This question will be answered here by using variable taxes. In contrast to the current tax law, where tax rates are fixed and exogenously determined, variable taxes are derived model-endogenously. Variable taxes are nonstochastic functions of capital market parameters to acquire new analytical solutions for asset pricing. In this way, equilibrium after taxes could be established, in case an equilibrium does not exist before taxes.

The following paper is organized as follows: In section 2, the standard model is documented on the basis of a complete capital market and the well-known CAPM framework. Section 3 formulates further assumptions about riskless lending and its taxation. Section 4 discusses a favorable tax on riskless assets. Using these prerequisites, we will then be prepared to define a variable tax rate on riskless assets and to deduce general equilibrium solutions for asset pricing in section 5. Section 6 states how to price options with respect to variable taxes. Section 7 summarizes the results, addresses the remaining open questions and offers prospects for further research.

## 2 The standard model

The premises of the classical CAPM of Sharpe (1964) and Lintner (1965) are well-known. They are as follows: Investors who have (1) a one period planning horizon, (2) are risk averse, (3) have homogenous beliefs and (4) decide ex ante upon the expectation and variance of future asset prices due to their individual utility function, which is strictly concave. The assumptions are:

(A1) There is a finite number of risky assets; short selling is allowed.
(A2) There is a riskless asset, which can be lent and borrowed without limits.

The rates of return on all risky assets are included in the stochastic n-vector **r**, where every single return $r_j$ $(j = 1, 2, \ldots, n)$ is a point in an n-dimensional vector space. A complete capital market has the following properties:[2] The first two moments of $r_j$ exist, $r_j \in H$ ,

$$H = \{ \sum_{j=1}^{n} \alpha_j r_j \mid \alpha_j \in R \} , \; H \subset L^2(\Omega, F, P) := \{ r_j : E(r_j^2) < \infty \} \;\; \forall \; j = 1, 2, 3, \ldots, n ,$$

the vector space H has dimension n, and the $r_j$ are linearly independent (i.e., **r** is a basis for H). The variance covariance matrix V**r** is symmetric, regular (i.e., invertible) and positive definite: the quadratic form $\mathbf{x}^T V\mathbf{r}\, \mathbf{x} > 0$ for $\mathbf{x} \neq \mathbf{0}$ (i.e., at least one element of $\mathbf{x} \in R^n$ must be nonzero, Bronstein et al., 2005). The first two moments of the distribution of **r** are given: E**r** is the vector of the expected rates of return and V**r** the associated variance covariance matrix. The rate of return $r_p$ on an arbitrary portfolio comprises n single assets,

$$r_p = \mathbf{w}^T \mathbf{r} , \tag{1}$$

with
$$Er_p = \mathbf{w}^T E\mathbf{r} , \tag{2}$$

$$Var(r_p) = \mathbf{w}^T V\mathbf{r}\, \mathbf{w} , \tag{3}$$

$$\mathbf{w}^T \mathbf{1} = 1 , \tag{4}$$

where **1** is an n-vector, which contains a column of 1´s. The nonstochastic vector $\mathbf{w} \in R^n$ contains the portfolio weights. The elements of **w** can be negative, because short sales are permitted. Because the matrix V**r** is positive definite as mentioned above, then the variance of the rate of return on an arbitrary portfolio (3) is always positive, even if assets are short sold.

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[2] Readers who are not well-versed mathematically may skip these technical requisites without great loss to the article´s general message.

What are the mean-variance optimal portfolios under A1 and A2? The optimization problem is

$$\min_{\{\mathbf{w}\}} \; (\mathbf{w}^{T} \cdot V\mathbf{r} \cdot \mathbf{w}) \tag{5}$$

for the constraint

$$\mathbf{w}^{T}\, E\mathbf{r} + (1 - \mathbf{w}^{T}\, \mathbf{1})\, r_f \;=\; Er_p \,. \tag{6}$$

The linear combination (6) indicates a portfolio, which consists partly of risky and riskless assets. The solution for (5) and (6) is carried out with the Lagrangian function according to Merton (1972). The solution of the optimization is the equation

$$\mu(r) \;=\; r_f \;\pm\; \sqrt{H}\, \sigma(r)\,, \tag{7}$$

where $\quad H = (E\mathbf{r} - r_f\,\mathbf{1})^{T}\, (V\mathbf{r})^{-1}\, (E\mathbf{r} - r_f\,\mathbf{1})\,.$ (8)

Equation (7) is the portfolio frontier under A1 and A2 (figure 1). “The portfolio frontier of all assets is composed of two half-lines emanating from the point $(0, r_f)$ in the $\sigma(r_p)$ - $E[r_p]$ plane with slopes $\sqrt{H}$ and $-\sqrt{H}$, respectively” (Huang and Litzenberger, 1988). According to the authors, a “portfolio is a frontier portfolio if it has the minimum variance among portfolios that have the same expected rate of return.”

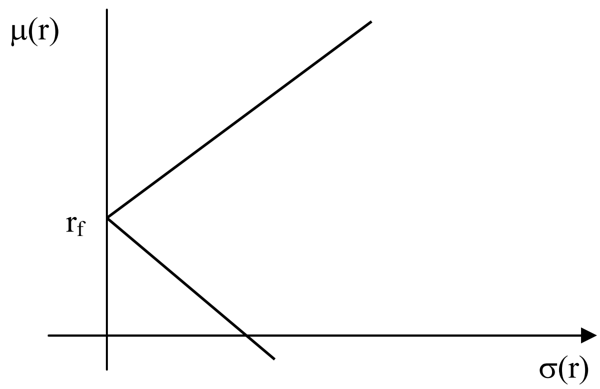


Figure 1: The portfolio frontier under A1 und A2.

Huang and Litzenberger (1988) describe in detail the condition under which investors would undertake risky investments if a riskless asset does exist according to the assumption A2. The location of the riskless rate compared with the hyperbolic portfolio frontier in the μ-σ-plane is decisive. According to the authors, risk averse investors will undertake risky investments if and only if the expected rate of return on the global minimum variance portfolio $Er_{mvp}$ exceeds the riskless rate,

$$Er_{mvp} > r_f \quad , \tag{9}$$

where $$Er_{mvp} = A/C \, , \tag{10}$$

$$A = \mathbf{1}^T (V\mathbf{r})^{-1} \, E\mathbf{r} \, , \tag{11}$$

$$C = \mathbf{1}^T (V\mathbf{r})^{-1} \, \mathbf{1} \, . \tag{12}$$

**Case 1:** If condition (9) is fulfilled, the half line (7) with positive slope is valid. In this case, there exists on the hyperbolic frontier exactly one efficient portfolio, the tangency portfolio (figure 2). All investors combine this portfolio with the riskless asset, allowing the possibility of short-selling the riskless asset and investing the proceeds in the tangency portfolio.[3] These two separating portfolios constitute the capital market line (CML):

> "If investors have homogenous beliefs, then they all have the same linear efficient set called the capital market line" (Copeland und Weston, 2004).

**Case 2:** If $Er_{mvp} < r_f$, the half line (7) with negative slope is valid and touches the hyperbolic frontier below $Er_{mvp}$ (figure 3). In this case, there exists not a single efficient portfolio on the hyperbola, so that all investors invest their entire wealth into the riskless asset. There exists only one efficient "portfolio", which consists of 100% of the riskless asset and of 0% of risky assets, depicted as a point $(0, r_f)$ in the $\mu$-$\sigma$-plane.

**Case 3:** If $Er_{mvp} = r_f$, the riskless rate lies in the vertex of the hyperbolic frontier (figure 4). A hyperbola has no tangency in the vertex, because the two asymptotes start from this point. In which case, a tangency portfolio does not exist and thus prompting all investors to invest exclusively in the riskless asset.

The authors argue case 2 involves short-selling the tangency portfolio and investing the proceeds in the riskless asset. Such a position is mathematically consistent with the constraint (6) and lies on the positively sloped half line (7) beyond the hyperbolic frontier. Because no risk averse investor buys risky assets, no market participant exists (bank, financial intermediary), who could take a long position in the tangency portfolio. Therefore, such positions have no economic meaning.

[3] These are the positions on the CML with higher returns than the tangency portfolio.

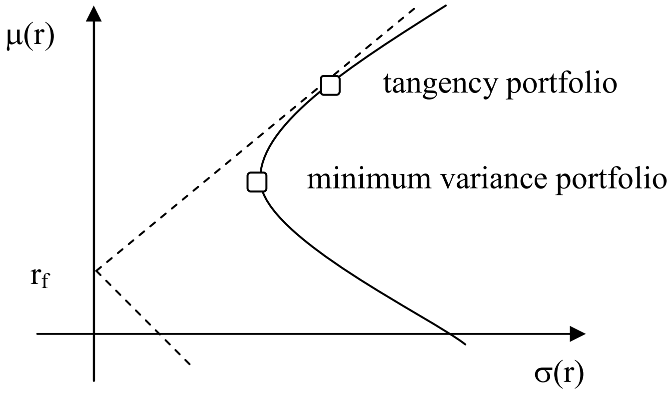


Figure 2: The portfolio frontier for $Er_{mvp} > r_f$ .

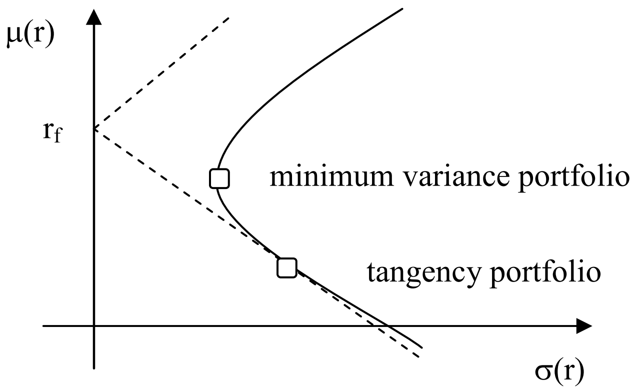


Figure 3: The portfolio frontier for $Er_{mvp} < r_f$ .

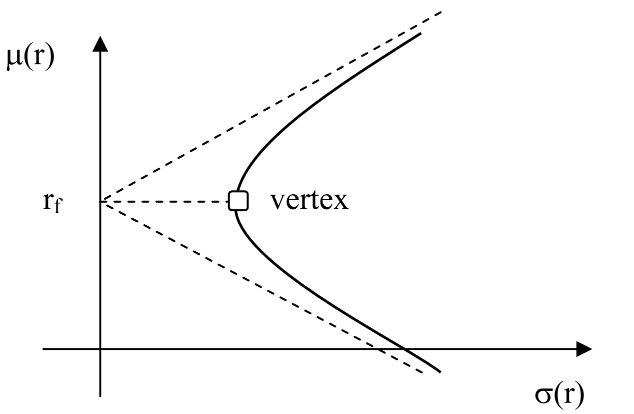


Figure 4: The asymptotes of the hyperbola ($Er_{mvp} = A/C = r_f$).

The existence of equilibrium and the validity of the CAPM is linked to the Huang-Litzenberger condition (9). Thus, the Huang-Litzenberger condition has the status of an equilibrium condition. If (9) is not fulfilled, a market portfolio does not exist because efficient portfolios do not exist.

> "Suppose that $r_f > A/C$. Then no investor holds a strictly positive amount of the market portfolio. This is inconsistent with market clearing. Thus in equilibrium, it must be the case that $r_f < A/C$ and the risk premium of the market portfolio is strictly positive" (Huang and Litzenberger, 1988).

Following the authors, equilibrium does not exist a priori. Therefore, the CAPM is not a general equilibrium model, at best it is only an equilibrium model under restrictions.

Whether or not the Huang-Litzenberger condition is fulfilled in real markets is an empirical issue. Both the prices of risky assets, which constitute the minimum variance portfolio, and the overnight rate are exogenous market variables. How they are related to each other is subject of empirical research.

## 3 Further assumptions

The CAPM is based on very simple assumptions. This has the advantage of a simple and clearly arranged model structure, but involves problems, which have already been mentioned above. Under A1 and A2, the CAPM cannot be qualified as a general equilibrium model. It would be desirable to keep the model as simple as possible and to make further assumptions in order to deduce equilibrium solutions for asset pricing, which are valid in general without any restriction. For this purpose, it suffices to modify the assumptions about risk-free lending and its taxation.

Until now the existence of a single riskless asset has been assumed under A2. This assumption will be modified here in the following way:

(A2*) There are several riskless assets. Short-selling the riskless asset is not allowed ("restricted borrowing").

Riskless rates are defined on

$$r_f \in [0, r_o] \quad , \tag{13}$$

where the overnight rate, $r_o$, is the maximal riskless rate an individual investor could obtain if he put his money into a riskless bank account. The overnight rate is assumed to always be greater than zero, $r_o > 0$, and can be represented globally by the London interbank offered rate, or Libor, reflecting the rate at which banks make short-term loans to one another. In Europe, the overnight rate can be represented by the Euro interbank offered rate, or Euribor. (13) defines $r_f$ as a compact interval and all possible riskless rates are included in it. The interval (13) is the realistic attempt, just as risky assets, to consider different riskless assets in the model, such as call money, deposit and current accounts.

The assumption A2* can be motivated both mathematically and economically. Mathematically infinite riskless rates can be considered. On a capital market, defined on $H \subset L^2(\Omega, F, P)$ with dimension n, these lie all on the hyper-line $\alpha \cdot \mathbf{1}$, where $\alpha \in R$, and **1** is an n-vector which contains a column of 1´s. Economically, such a line does not make any sense. One has to shrink the possible set of riskless rates while restricting the line, for example to a single point according to A2, $\alpha = r_o$, or to a line between two points according to A2*, $\alpha \in [0, r_o]$. Economically, A2* makes sense and is realistic because, on real markets, riskless assets exist with different rates. For example, a deposit account could have a rate between zero and the overnight rate. The intuition is that riskless assets exist which are more liquid (fungible) than others so that the investor can act more quickly and unbureaucratically upon the money. Also, an investor has the ability to invest a part of his wealth interest-free, $r_f = 0$, or to hold cash in his portfolio. However, with A2* sub-optimal risk-free lending is in principle allowed and possible for investors. Of course no rational investor would choose a riskless asset with a sub-optimal interest rate. Therefore, the portfolio selection for (13) is the same as under A1 and A2. The question arises: What is the sense in A2*, if it does not essentially alter the portfolio selection? The real meaning of A2* emerges in connection with a worst case scenario, which is described later in section 4.

Furthermore, under A2*, “Investors are not allowed to take short positions in the riskless assets” (Black, 1972). This restriction can be stated less strictly by the term “constrained borrowing”, which means short-selling the riskless asset at a higher rate than the overnight rate (i.e., a spread emerges between borrowing and lending). This becomes relevant if riskless assets are taxed. Even if constrained borrowing were more realistic, restricted borrowing according to A2* will be assumed here because this allows a simpler modelling. It can easily be shown that the result concerning the efficient set does not alter very much if constrained borrowing is assumed. However, A2* serves here first and foremost a technical purpose, which evolves later in connection with the taxation of riskless assets (assumption A3).

What is the portfolio selection under A2*? The Lagrangian approach under A1 and A2* is analogous to A1 and A2 and (5) and (6) for the additional constraint

$$1 - \mathbf{w}^{T}\, \mathbf{1} \geq 0\ . \qquad (14)$$

In this case the Lagrange function becomes more awkward and involved:

> "Unfortunately, the method for finding the constrained maxima in problems with inequality constraints is a bit more complex than the method we used for equality constraints. The first order conditions involve both equalities and inequalities and their solution entails the investigation of a number of cases" (Simon and Blume, 1994).

Instead, one can use graphic solutions: geometrically, the CML under A2* is a line between two points starting at the point $(0, r_o)$ and ending in the tangency point $(\sigma(r_t), Er_t)$. The CML is bound on the right hand side and the mean values lie within the interval $[r_o, Er_t]$. Investors choose a convex combination[4] of the tangency portfolio and the riskless asset with the maximum rate (this is the overnight rate), which constitutes the two separating portfolios (figure 5).

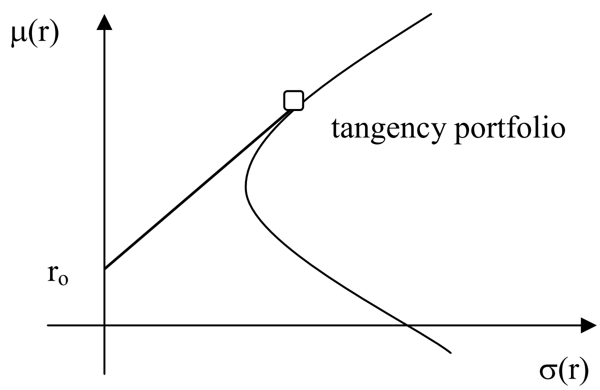


Figure 5: The CML under restricted borrowing.

Finally, what are the mean-variance efficient portfolios under A1 and A2*? Under A2* there are two possible sets of efficient portfolios: on the one hand, portfolios which lie on the CML defined above and, on the other hand, portfolios which lie on the upper

[4] A convex function has the representation $z = \lambda\, x + (1 - \lambda)\, y$ for $x,y \in R$ and $\lambda \in [0, 1]$, see Bronstein et al. (2005).

branch of the hyperbolic frontier and have higher returns than the tangency portfolio. The latter represent portfolios which comprise exclusively risky assets (Black, 1972).

Additional to A1 and A2* we suppose:

(A3) riskless assets are flat taxed.

Under A3, the taxation is effected by a given tax rate, which is valid for all investors, independent of their income. Therefore, all investors face the same riskless rates after taxes.

How does A3 influence the portfolio selection? Because a flat rate maintains the decision rule "homogenous beliefs", the CAPM after taxes can be deduced as simply as before taxes. The optimization under A1, A2* and A3 can be done analogously to A1 and A2*, with the difference that the maximal riskless rate after taxes, $r_{f,at,max}$, replaces the riskless rate (before taxes). Under A1 – A3 one obtains the after-tax version of the Zero-Beta CAPM according to Black (1972),

$$Er_j = Er_{z(m)} + \beta_{jm} (Er_m - Er_{z(m)}) \tag{15}$$

for $$Er_{z(m)} \geq r_{f,at,max} , \tag{16}$$

where $r_j$ is the rate of return on an arbitrary risky asset, $r_m$ is the return on the market portfolio, $r_{z(m)}$ is the return on the corresponding zero covariance portfolio and $\beta_{jm}$ is the well known β-factor. The validity of (15) is linked to the condition (16), which is the efficiency condition of the market portfolio with respect to the maximal riskless rate after taxes.

It can easily be shown that taxes on risky assets, taxes on dividends and capital gains do not essentially alter the result, assuming they are also flat taxed and therefore independent of the individual income. According to most tax laws, stocks are double taxed, once on the corporation and once again at the stockholder. The double taxation of stocks is controversial and will not be discussed here.

## 4 Profit versus wealth tax

The assumption A3 says nothing about the tax rate to tax riskless assets. According to tax in law,

$$r_{f,at} = (1 - \tau_f)\, r_f \qquad \text{for} \quad r_f \in [0, r_o] , \quad \tau_f \in [0, 1) , \tag{17}$$

where $\tau_f$ is the profit (yield) tax rate while $r_{f,at}$ represents all possible riskless rates after taxes. The maximal riskless rate after taxes is

$$r_{f,at,max} = r_{o,at} = (1 - \tau_f)\, r_o\,,$$

which is called here the "overnight rate after taxes". This is the maximal riskless rate of return, which an individual investor could obtain after taxes if he put his money in a riskless bank account. A wealth tax does not conform to the current tax law and represents a second possibility of taxing riskless assets. The equations to define a wealth tax on riskless assets are

$$W_{1,at} = (1 - \upsilon_f)\, W_1\,, \qquad W_1 > 0\,,\ \ \upsilon_f \in [0, 1)\,, \tag{18}$$

$$W_1 = W_o\,(1 + r_f)\,, \qquad W_o > 0\,, \tag{19}$$

$$W_{1,at} = W_o\,(1 + r_{f,at})\,. \tag{20}$$

$W_o$ is the nonstochastic riskless invested wealth and $W_1$ and $W_{1,at}$ are the riskless wealth at the end of one period (one year) before and after taxes respectively. From (18) – (20) we get all possible riskless interest rates after taxes,

$$r_{f,at} = (1 + r_f)\,(1 - \upsilon_f) - 1 \qquad \forall\ r_f \in [0, r_o]\,, \tag{21}$$

and the overnight rate after taxes,

$$r_{o,at} = (1 + r_o)\,(1 - \upsilon_f) - 1\,, \quad \text{if } r_f = r_o\,. \tag{22}$$

We have to check, despite the current tax law, which tax rate is favourable with respect to capital market equilibrium. For the time being, both possibilities, the profit and the wealth tax, should be kept open. It turns out that only the latter allows the deduction of general and unrestricted equilibrium solutions for asset pricing.

It is shown that it is not possible to deduce general equilibrium solutions with a profit tax (17) according to current tax laws. The Huang-Litzenberger condition after taxes,

$$Er_{mvp} > r_{o,at}\,, \tag{23}$$

implies that the existence of equilibrium depends also on the tax rate.[5] In the case of a profit tax, there is an interval of favourable tax rates and, inside this, equilibrium exists,

$$1 - \frac{Er_{mvp}}{r_o} < \tau_f < 1 \qquad \text{for } \ Er_{mvp} \in (0, r_o]\,,\ \ \tau_f \in [0, 1)\,. \tag{24}$$

(24) illustrates, which profit tax is favourable with respect to the Huang-Litzenberger condition after taxes (23). A tax rate is favourable if it lies inside the interval (24), but is

[5] If risky assets are also taxed, we get $Er_{mvp,at} > r_{o,at}$.

otherwise unfavourable. In this way, the tax rate $\tau_f$ can be checked with respect to equilibrium. The interval (24) ensures an equilibrium after taxes, but only for $Er_{mvp} \in (0, r_o]$ and one period. The worst case

$$Er_{mvp} \leq 0 \tag{25}$$

is not yet considered. In the worst case, (25), all riskless assets including the interest-free asset (cash) dominate the return expectation of risky investments. In order to prevent every investor from putting his money into a riskless bank account, a very special taxation of riskless assets seems to be necessary. Obviously, it must be

$$r_{o,at} < Er_{mvp} \leq 0 \ . \tag{26}$$

According to (26), all riskless interest rates must be negative after taxes to make equilibrium after taxes possible. Most importantly, interest-free assets have to be taxed to give risky assets a chance to build demand. This cannot be done with a profit tax where all possible riskless rates after taxes lie within the interval $(0, r_o)$. Therefore, a profit tax is not helpful in obtaining general equilibrium.

This should be motivation to introduce a wealth tax on riskless assets according to (18) – (22). Such a tax makes it possible to deduce equilibrium solutions, including the worst case. Obviously, under worst case conditions (25), it must be:

$$\upsilon_f > \frac{r_o}{1 + r_o} \ . \tag{27}$$

Putting (27) into (22) results in a negative overnight rate after taxes,

$$r_{o,at} < 0 \ .$$

Thus (27) presupposes that the Huang-Litzenberger condition after taxes (23) is fulfilled in the worst case (25), such that equilibrium exists in any case.

A wealth tax on riskless assets is a very new and unfamiliar affair. To illustrate how riskless rates can be modelled with a wealth tax, (21) is written up in the following way:

$$r_{f,at} = f(r_f, \upsilon_f) = r_f - (1 + r_f)\, \upsilon_f \ . \tag{28}$$

Figure 6 shows equation (28) for different riskless assets. A remarkable feature of the new tax is that interest rates can become negative after taxes, $r_{f,at} \in (-1, r_o)$. This is the case for

$$\upsilon_f > \frac{r_f}{1 + r_f} \qquad \forall \ \ r_f \in [0, r_o] \ .$$

Negative interest rates are, as mentioned above, the precondition for equilibrium solutions including the worst case (25). Another characteristic of a tax on riskless assets is the taxation of interest-free assets (e.g. cash). In this case, (21) is reduced to

$$r_{f,at} = -\upsilon_f \qquad \text{if} \quad r_f = 0\,. \tag{29}$$

(29) says the interest rate on cash after taxes corresponds to the negative tax rate. Thus, if the tax rate is positive, $\upsilon_f > 0$, the interest rate on cash after taxes is always negative.

Money in the form of cash, call money or current accounts, is primarily used for payment transactions and does not represent a riskless asset. Current accounts are used for payments as long as the deposited amount, in the case of a private investor, does not exceed two to three monthly salaries. Current accounts can be used partially for transactions and partially for riskless lending. The latter represents a riskless asset and should be taxed as such. However, if such accounts will be taxed, a tax allowance must be granted. Therefore, money that is used for payment transactions remains untaxed.

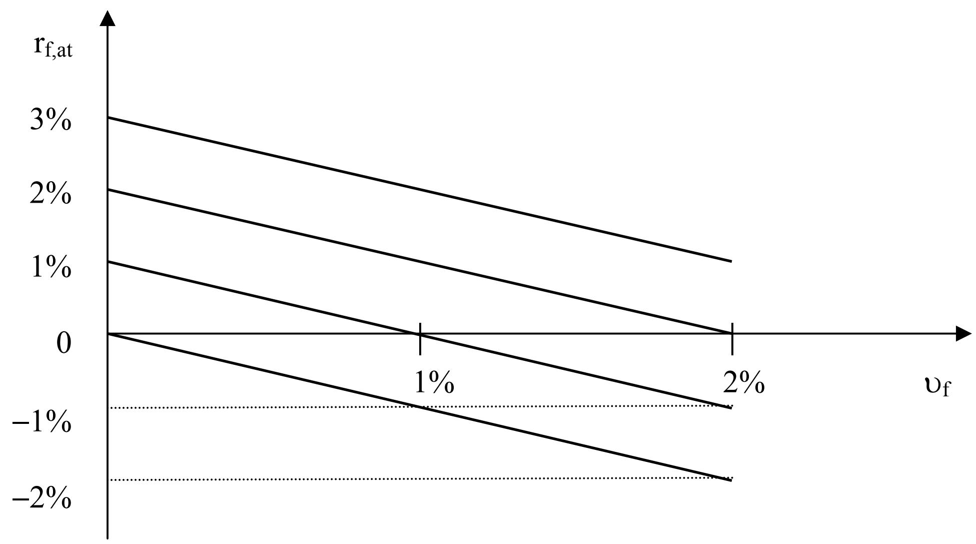


Figure 6: Interest rates after taxes for $r_f = 0$, $r_f = 1\%$, $r_f = 2\%$ and $r_f = 3\%$, when riskless assets are taxed with a wealth tax.

## 5 Deduction of the CAPM after variable taxes

On the basis of the extended assumptions A1 – A3, a general equilibrium theorem is formulated.

**Theorem:** Under A1 – A3 the following assertions are equivalent:

(1) A general and unrestricted capital market equilibrium exists.

(2) There is a value $g_o \in (-1, r_o)$ with the following properties:

$$g_o = r_{f,at,max} \quad \text{and} \quad g_o < Er_{mvp} . \tag{30}$$

(3) Asset pricing is independent of the overnight rate.

**Proof:**

(1) ↔ (2) The assertions (1) und (2) are obviously equivalent because equilibrium only exists if the Huang-Litzenberger condition after taxes (23) is fulfilled and vice versa.

(2) → (3)[6] (a) By contradiction:[7] Assuming equilibrium and a function $g_o^* = f(r_o)$ according to (30), which depends on $r_f$. Equilibrium implies that $g_o^* = r_{f,at,max} = a \cdot Er_{mvp}$ for $a \in (0, 1)$, so that (23) is fulfilled. If $g_o^* = a \cdot Er_{mvp}$, then $g_o^*$ can not be a function of $r_o$, because $Er_{mvp}$ is, according to (10) – (12), exogenous and "a" is an arbitrary number between zero and one. This is a contradiction to the above assumption, therefore $g_o \neq f(r_o)$.

(b) The Zero-Beta CAPM (15) is independent of $r_o$ if $g_o$ is put into (16) instead of $r_{f,at,max}$. Because $g_o \neq f(r_o)$, also $Er_j \neq f(r_o)$ .

(3) → (2) The CAPM is independent of $r_o$ if $r_o$ does not exist. Thus, it suffices to show that under A1 (without A2) a general equilibrium does exist (according to Black, 1972, or Hens, Laitenberger and Löffler, 2002). □

The theorem states that in equilibrium an unknown value $g_o$ exists with certain mathematical properties, which guarantees general equilibrium and is independent of interest rates before taxes. The value $g_o$ is not yet linked with a real economy, because $g_o$ was until now a purely hypothetical value. The theorem names in point (2) are solely necessary conditions on $g_o$. The absolute value of $g_o$ is still unknown.

[6] Because (1) and (2) are contingent upon one another, it suffices to show, that (3) follows from (2) and vice versa.

[7] The proof is carried out only for $Er_{mvp} > 0$ and can be done analogously for $Er_{mvp} \leq 0$.

The following assumption helps us implement the hypothetical value $g_o$ in a real economy.

(A4) Riskless assets are variably taxed.

In connection with A4, a variable tax rate on riskless assets can be defined. Within a mean-variance framework, this could be done with a nonstochastic function of capital market parameters, the overnight rate, and some constants,

$$\upsilon_f = f(E\mathbf{r}, V\mathbf{r}, r_o, c_1, c_2, \ldots), \qquad \upsilon_f \in [0, 1) . \tag{31}$$

Because capital market parameters are not constant over time, the tax rate (31) can take different values in different periods. The length of a period is undetermined. The question arises, which function is appropriate and makes sense in this context? The following proposition provides a model-endogenous definition for a particular family of variable tax rates.

**Proposition 1:** Under A4 and with the tax rate

$$\upsilon_f = \frac{r_o - g_o}{1 + r_o} \qquad \text{for} \qquad g_o \in (-1, r_o), \tag{32}$$

the following equation holds:

$$g_o = r_{o,avt} = r_{f,at,max} , \tag{33}$$

where $r_{o,avt}$ represents the overnight rate after variable taxes.

**Proof:** Rearranging (32) leads to $g_o = (1 + r_o)(1 - \upsilon_f) - 1 = r_{o,at}$, which is identical with equation (22). □

Because of (33), the proposition 1 provides a financial realization of the unknown value $g_o$ and therefore an economic meaning of the theorem.

In the following, $g_o$ will be determined in capital market equilibrium. Looking for an equation or condition to determine $g_o$, one could recall the existence of a market portfolio in equilibrium. Therefore, in equilibrium, a further condition of $g_o$ is known: the efficiency condition for the market portfolio (16). This condition compares the market portfolio (its zero covariance portfolio) with the maximal riskless rate after taxes. Therefore, we obtain also a solution for $g_o$. This solution will be assured by the following proposition, starting from a portfolio, which is efficient under A1 (i.e., located on the upper branch of the hyperbolic frontier). This portfolio does not need to be efficient under the additional assumption A2 (or A2*).

**Proposition 2:** Given an arbitrary portfolio “q”, which is efficient under A1 (without A2), then

$$g_o \leq Er_{z(q)} \tag{34}$$

provides under A1 – A4 a necessary and sufficient condition for equilibrium.

**Proof:** Because $g_o = r_{f,at,max}$ , (34) ensure the efficiency of the portfolio “q” under A1 – A4 and therefore equilibrium after taxes. □

The proposition 2 is true because of the assumption A3 and A4 concerning the taxation of riskless assets, which ensures that all investors calculate with the same riskless rate after (variable) taxes. Proposition 2 provides a sufficient condition and thus an equilibrium solution for $g_o$.

With the propositions 1 and 2, it takes only a small step to deduce the CAPM after variable taxes. Starting again from an arbitrary portfolio “q” on the upper branch of the hyperbolic frontier and its zero covariance portfolio, the value $g_o$ can be chosen as follows:

$$g_o = r_{o,avt} = Er_{z(q)} \qquad \text{for} \quad g_o \in (-1, r_o) \, . \tag{35}$$

Figure 7 illustrates equation (35) in the μ-σ-plane. Due to proposition 2, the portfolio “q” need not be efficient before taxes. Therefore,

$$Er_{z(q)} < r_o \tag{36}$$

is allowed. According to proposition 2 the portfolio “q” is mean-variance efficient after taxes and this results in the formula

$$Er_j \; = \; Er_{z(m)} + \beta_{jm} \, (Er_m - Er_{z(m)}) \tag{37}$$

for $\qquad Er_{z(m)} \geq r_{o,avt}$ . (38)

The formulas (37) and (38) represent the Zero-Beta CAPM after variable taxes. The β-factor is the same as in the classical CAPM. Unlike the after-tax-version (15) and (16), the CAPM after variable taxes according to (37) and (38) is always valid, because equilibrium is guaranteed through (30) and (35). Furthermore, (37) is independent of the overnight rate, which has been shown by the theorem. The overnight rate is still relevant for calculating the variable tax rate (32) but not in the CAPM after variable taxes (37). Therefore, asset pricing after variable taxes is completely independent of interest rates before taxes.

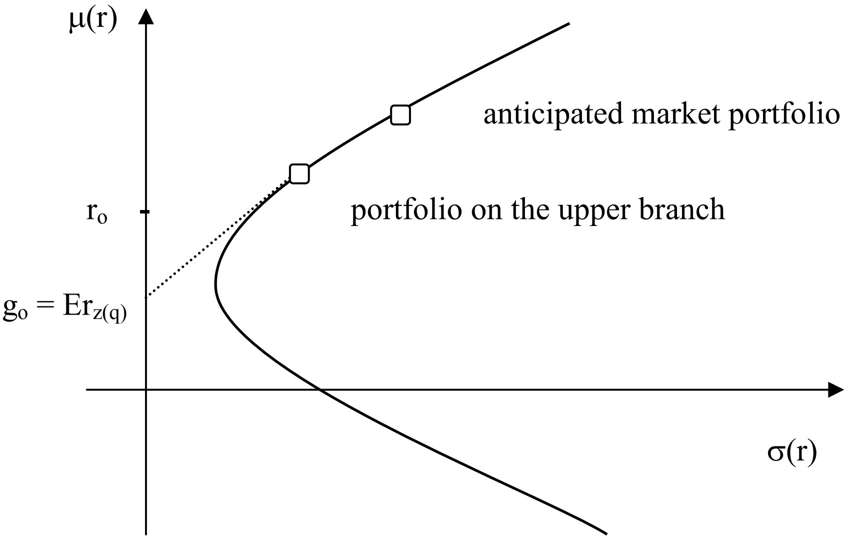


Figure 7: Anticipated equilibrium after variable taxes.

The equilibrium solution (37) and (38) is, for the time being, a purely analytical result. How can it be applied to price risky assets in practice? The formula (37) and (38) can be simplified in the following way: It is assumed that all investors hold portfolios which lie on the CML after taxes. This implies that the chosen portfolio "q" anticipates the market portfolio,

$$r_q \equiv r_m \,. \tag{39}$$

In this case the CAPM after variable taxes is

$$Er_j \;=\; r_{o,avt} + \beta_{jm}\,(Er_m - r_{o,avt}) \tag{40}$$

due to the classical CAPM of Sharpe (1964) and Lintner (1965). The portfolio "q" can be called "anticipated market portfolio" because of (39), which need not be efficient before taxes, but after taxes. This implies that this portfolio comprises (nearly) all risky assets of an economy according to a broadly diversified share index. The identity (39) confronts the critical reader with an insoluble paradox in the CAPM that portfolios on the hyperbolic frontier do include short-selling of risky assets, but the market portfolio does not (Ross, 1977).

Equation (40) can still be simplified. Starting from (22), the overnight rate after variable taxes can be reduced to

$$r_{o,avt} \;\approx\; r_o - \upsilon_f \;, \tag{41}$$

if the product $r_o \cdot \upsilon_f$ is neglected. The intuition is that the tax rate $\upsilon_f$ involves an artificial inflation on the overnight rate. Inserting (41) into the CAPM after variable taxes (40) provides

$$Er_j \approx r_o - \upsilon_f + \beta_{jm} (Er_m - r_o + \upsilon_f) . \quad (42)$$

In (42) the variable $\upsilon_f$ can be interpreted as a control variable, which is a function of exogenous financial parameters to stabilize financial markets: If share prices rise, $\upsilon_f$ is low and riskless assets are taxed moderately. When share prices stagnate, $\upsilon_f$ is high and riskless assets are taxed more strongly in order to give risky assets a chance to recover. This regulatory property of the new tax is particularly remarkable.

## 6 Option pricing after variable taxes

Dybvig and Ingersoll (1982) put into concrete terms what kind of assets can be priced with the CAPM. According to the authors, mean-variance asset pricing is valid for the primary assets, which are essentially common stocks. Financial assets, represented by future contracts, options, and securities of financial intermediaries cannot be priced within the CAPM. As soon as financial assets are considered in the CAPM, it turns out that arbitrage possibilities arise.

> "We argue that this result is a broadly applicable justification for ignoring financial assets when applying the CAPM" (Dybvig and Ingersoll, 1982).

Therefore, option pricing has to take place beyond the CAPM. It is well-known that options are priced in models especially elaborated for options and other derivatives (Björk, 1998, or Korn and Korn, 2001).

The CAPM is based on equilibrium considerations, while option pricing is based on arbitrage considerations. One could ask, "How important is equilibrium on capital markets for option pricing?" Primary assets constitute the underlying for options, and only if the underlying has a positive price can options be replicated. Therefore, capital market equilibrium is also a precondition for option pricing.

It is shown in the previous sections that equilibrium solutions for primary assets do not exist a priori, but depend on the Huang-Litzenberger condition (9) and (23) respectively. It is further shown in section 5 that equilibrium solutions are linked to an artificial value $g_o$ with well-defined mathematical properties. It is also shown that $g_o$ is suitable to define a variable tax rate on riskless assets. According to equation (33), $g_o$ can be identified with the overnight rate after variable taxes $r_{o,avt}$, which is exclusively a function of capital market parameter and independent of the overnight rate (before taxes). Finally, in a general mean-variance equilibrium framework, $r_{o,avt}$ becomes relevant for asset pricing instead of the overnight rate.

How can one price options if riskless assets are variably taxed? Until now, options are priced on a complete capital market without taxes and other institutional restrictions on the basis of the overnight rate. If there exists a variable tax on riskless assets due to proposition 1, options have to be priced on the basis of $g_o$ due to (35). The algorithm of option pricing is the same and can be applied according to the *fundamental theorem of asset pricing* of Delbaen and Schachermayer (1999). The only difference is the discount factor: Instead of the overnight rate one has to calculate with the overnight rate after variable taxes, which is a function of capital market parameters.

## 7 Conclusion

It is shown that further assumptions about risk-free lending and its taxation suffice to deduce general and unrestricted equilibrium solutions for mean-variance asset pricing. Therefore, it is assumed that various suboptimal riskless assets exist beside the overnight rate. It is also assumed that riskless borrowing is restricted, and lastly, that riskless assets are flat taxed (i.e., the tax rate does not dependent on the individual´s income). On the basis of these additional assumptions, an equilibrium theorem is formulated which involves necessary conditions for general equilibrium in the sense of Huang and Litzenberger (1988). The sufficient condition is provided by a variable tax rate on riskless assets, which is defined endogenously as a function of capital market parameters. The variable tax rate contains all relevant information to ensure equilibrium after taxes. The result is an asset pricing formula, denoted as CAPM after variable taxes, which has the same form as the Zero-Beta CAPM of Black (1972). The decisive difference between the CAPM after variable taxes and the version of Black is that the former is not conditioned by the overnight rate (before taxes). Thus, asset pricing after variable taxes depends exclusively on capital market parameters and is independent of the overnight rate.

One question remains unanswered, that of how to tax bonds if riskless assets are variably taxed. It seems reasonable also to tax bonds variably, but not with the same tax rate as riskless assets, because bonds have a price risk the longer the maturity lasts.

Possible is a variable tax rate on bonds, which is also a function of the maturity of the bond. This question is left for further research to develop an adequate model.

The new tax has a regulatory impact. Especially in time periods of high interest rates and low yields on equity, a variable tax on riskless assets (and bonds) can compensate for stagnation especially on stock markets. While taxing riskless assets more strongly, there is more scope for the firms to consolidate their profits and to attract potential investors, the relevance of which need not be highlighted given the current financial market situation.

## Appendix

The following formulas are helpful to understand the CAPM (proofs see Huang and Litzenberger, 1988).

**Proposition A.1:** Under A1 every portfolio "q" on the hyperbolic frontier (except the minimum variance portfolio) has a corresponding zero covariance portfolio "z(q)", which satisfies

$$Er_{z(q)} = \frac{A}{C} - \frac{D / C^2}{Er_q - A / C} ,$$

$$B = (E\mathbf{r})^T (V\mathbf{r})^{-1} E\mathbf{r} ,$$

$$D = BC - A^2 ,$$

A and C according to (11) and (12).

**Proposition A.2:** Under A1 the following formula holds for an arbitrary risky asset "j" and an arbitrary portfolio on the hyperbolic frontier (except the minimum variance portfolio),

$$Er_j = \beta_{jq} Er_q + (1 - \beta_{jq}) Er_{z(q)} \qquad \forall \; j = 1, 2, ..., n ,$$

$$\beta_{jq} = Cov(r_j, r_q) / Var(r_q) ,$$

which is "A mathematical fact … without using economic reasoning" (Huang and Litzenberger, 1988).

**Proposition A.3:** Under A1 and A2, the tangency portfolio "t" fulfils $Er_{z(t)} = r_f$ if $Er_{mvp} > r_f$ .

## Literature

Björk, T. (1998). *Arbitrage theory in continuous time*. Oxford University Press.

Black, F. (1972). Capital market equilibrium with restricted borrowing. *Journal of Business, 45*, 444-455.

Brennan, M. J. (1970). Taxes, market valuation and corporate financial policy. *National Tax Journal, 13, 4,* 417-455.

Brockwell, P. J., and Davis, R. A. (1991). *Time series, theory and methods*. New York: Springer.

Bronstein, I. N. et al. (2005). *Taschenbuch der Mathematik*. Frankfurt: Harri Deutsch.

Copeland, T. E., and Weston, J. F. (2004). *Financial theory and corporate policy*. Boston: Pearson Addison Wesley.

Delbaen, F., and Schachermayer, W. (1998). The fundamental theorem of asset pricing for unbounded stochastic processes. *Mathematische Annalen, 312*, 215-250.

Duffie, D. (1990). *Security markets*. Boston Academic Press.

Dybvig, P. H., and Ingersoll, J. E. Jr. (1982). Mean-variance theory in complete markets. *Journal of Business, 55*, 233-251.

Hens, T., Laitenberger, J., and Löffler, A. (2002). Two remarks on the uniqueness of equilibria in the CAPM. *Journal of Mathematical Economics, 37*, 123-132.

Huang, C.-F., and Litzenberger, R. H. (1988). *Foundations for financial economics*. New York: North-Holland.

Karr, A. F. (1993). *Probability*. New York: Springer.

Kandel, S., and Stambaugh, R. F. (1987). On correlations and inferences about mean-variance efficiency. *Journal of Financial Economics, 18*, 61-90.

Korn, E., and Korn, R. (2001). *Optionsbewertung und Portfolio-Optimierung*. Wiesbaden: Vieweg.

Kruschwitz, L., and Schöbel, R. (1987). Die Beurteilung riskanter Investitionen und das Capital Asset Pricing Model (CAPM). *Wirtschaftswissenschaftliches Studium*, *Heft 2*, 67-72.

Lintner, J. (1965). The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *Review of Economics and Statistics, 47*, 13-37.

Litzenberger, R. H., and Ramaswamy, K. (1979). The effect of personal taxes and dividends on capital asset prices. *Journal of Financial Economics, 7*, 163-195.

Markowitz, H. (1952). Portfolio selection. *Journal of Finance, 12*, 77-91.

Merton, R. C. (1972). An analytical derivation of the efficient portfolio frontier. *Journal of Financial and Quantitative Analysis, 7*, 1851-1872.

Ross, S. A. (1977). The Capital Asset Pricing Model (CAPM), short sale restrictions and related issues. *Journal of Finance, 32*, 177-183.

Schreiber, U., Spengel, C., and Lammersen, L. (2002). Measuring the impact of taxation on investment and financing decisions. *Schmalenbach Business Review, 54*, 2-23.

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance, 19*, 425-442.

Simon, C. P., and Blume, L. (1994). *Mathematics for economists*. New York: Norton.

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